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Bayesian-Bootstrap Loss Development

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Bayesian Framework

Model	Example	Analogy
Unobservable Qty	Claim Freq θ	Ultimate Loss U
Prior Distribution	$\theta \sim \Gamma(\alpha,\beta)$	$U \sim g_U(u)$
Observable Qty	Number of claims N	Loss at n^{th} report L
Probabilistic Model	$N \mid \theta \sim \text{Poisson}(\theta)$	$L \mid U \sim U / \Lambda$ where Λ = FTU*
Observation	$N = n$	$L = l$
Posterior Distribution	$\theta \sim \Gamma(\alpha+n,\beta+1)$	$U \sim$ new $g_U(u)$
Predictive Distribution	$N \sim$ Negative Binomial	$L \sim h(l)$

* FTU=Factor-to-Ultimate

What do we need to apply the model?

Prior for ultimate	$U \sim g_U(u)$
Observed loss given ultimate	$L \mid U \sim U / \Lambda$
Distribution of FTU	$\Lambda \sim g_\Lambda(\lambda)$
Conditional dist'n of FTU	$\Lambda \mid U \sim g_\Lambda(\lambda \mid U)$

- Prior distribution of ultimate losses
  - Computation of aggregate losses now standard
    - FFTs, Heckman-Meyers, Method of Moments
    - There are no others...
- Distribution for FTUs using bootstrap
- Essential ingredient: joint distribution of *U* and FTU

$$g(\lambda, u) = g_\Lambda(\lambda \mid U)\, g_U(u)$$

Ingredients

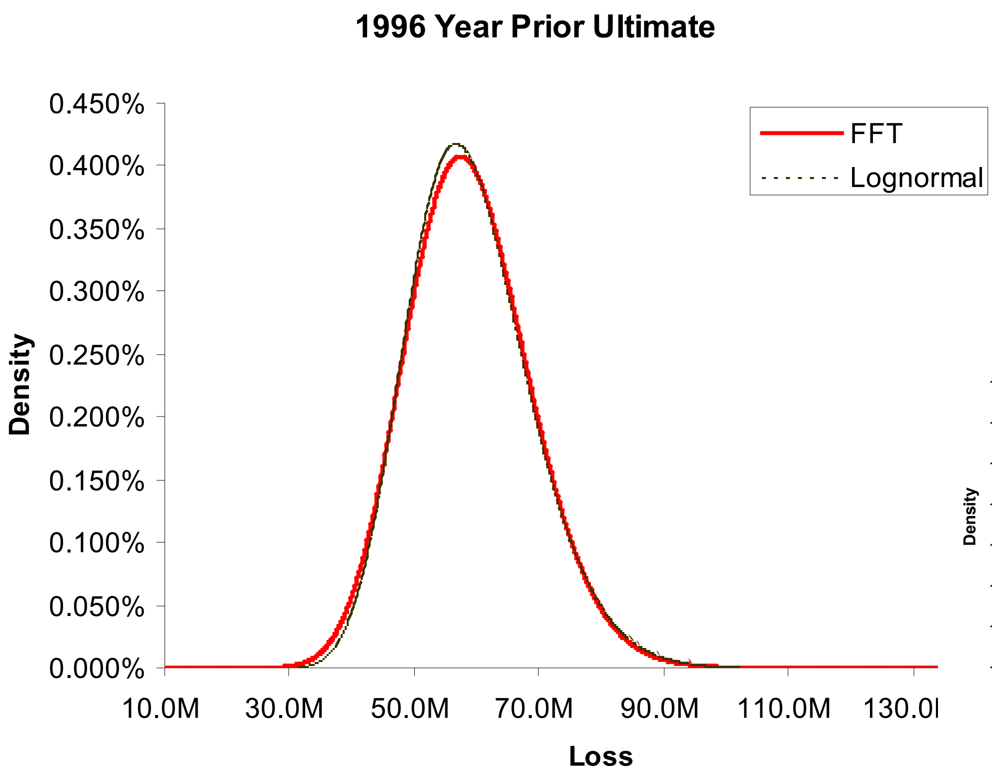
Parametric and Non-Parametric Distributions

- Predilection for parametric distributions
- Computers make non-parametric, numerical, discrete distributions easy to use
- Offer great flexibility: capture cluster points
- No tricky fitting problems
- Produced by cat models
- Easy to compute statistics, layers, etc.
- Appeal of parametric distributions driven by lack of powerful computers!

Using Fast Fourier Transforms to Compute Aggregate Distributions

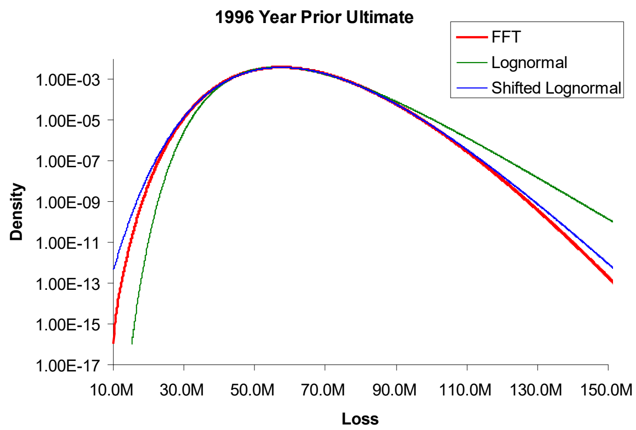
- Fast and efficient method
- Clearly explained in Wang [9]
- Easy to code in Excel
- Use VBA functions, not IMPRODUCT spreadsheet functions
- Can code FFT in VBA based on Numerical Recipies algorithms [6]
- Alternatively, can link to DLLs
  - See Solomon [7] for method
  - See Intel web page [4] for free DLLs
- FFT of real vector is conjugate symmetric
  - Halves needed computations

Prior Ultimate Loss Distribution



FFT generated aggregate

Lognormal approximations fitted using method of moments



- Mean: $58.9M
- CV: 0.168
- Skew: 0.307

Freq: Negative Binomial

Contagion ~ 0.02

Severity: 5 Param Pareto

Example

Favorite Method

- Lognormal link ratios
  - Product of lognormals is lognormal
- No other reason?

Bootstrap Method

- Link ratios in triangle with *n* years data can be re-sampled to give (*n*–1)! different FTUs
  - 9! =362,880; 17! = 355,687,428,096,000
- Bootstrapping explained in Ostaszewski Forum article [5] and Efron and Tibshirani book [1]

7 Example

Link Ratios	1 to 2	2 to 3	3 to 4	4 to 5	5 to 6	6 to 7
1991	4.155	1.579	1.796	1.290	1.197	1.123
1992	3.359	1.417	1.644	1.300	1.247	
1993	3.593	1.752	1.977	1.434		
1994	4.567	1.547	1.838			
1995	1.920	1.670				
1996	4.529					

Resamples

1	3.359	1.417	1.644	1.434	1.197	1.123
2	1.920	1.670	1.644	1.434	1.247	1.123
3	4.567	1.752	1.977	1.300	1.247	1.123
	…	…	…	…	…	…

FTU

1	15.097	4.495	3.172	1.929	1.345	1.123
2	10.593	5.516	3.302	2.008	1.400	1.123
3	28.806	6.307	3.599	1.820	1.400	1.123
	…	…	…	…	…	…

Advantages of Bootstrap

- Relies on available data
- Quick and easy to code
- No need to make questionable assumptions on link ratio distribution
- No need for complex curve fitting
- Method gives payout pattern and distribution of discount factors
- Produces confidence intervals around estimates

Bootstrap FTUs

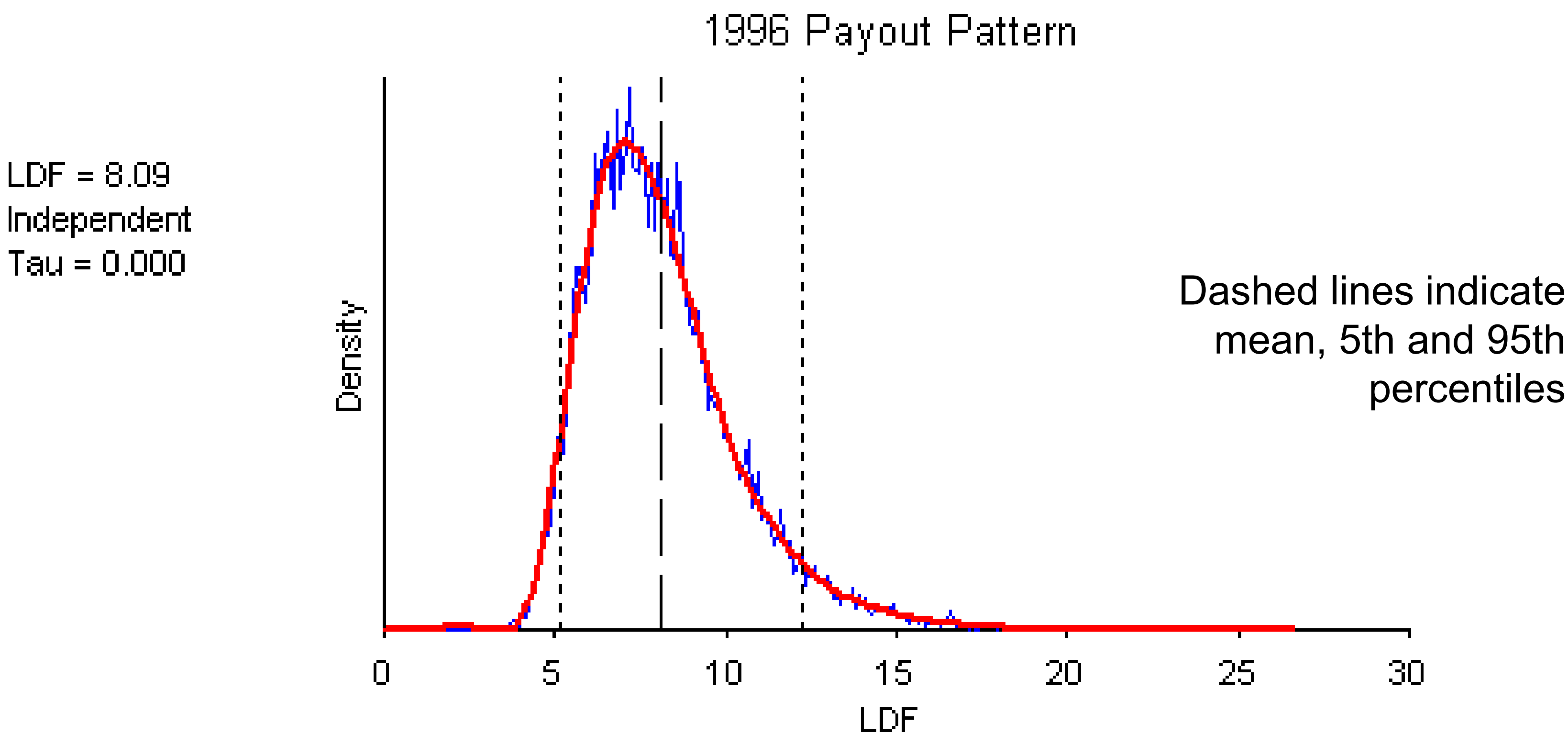
Ah but…

- What about inflation and other unique historical episodes in data?
- What about correlation between first two link ratios?
- What about the re-engineered claims department, changes in reserving, tort reform, social inflation, Y2K liability?
- No data, small triangle?

Try

- Triangle must be adjusted for perceived anomalies
- Bootstrap techniques available to retain correlation structure; re-sample in pairs
- Same problems exist for traditional applications of triangles. Use same solutions!
- Combine triangles, use similar LOB, and other methods used for reserving

Distribution of FTUs

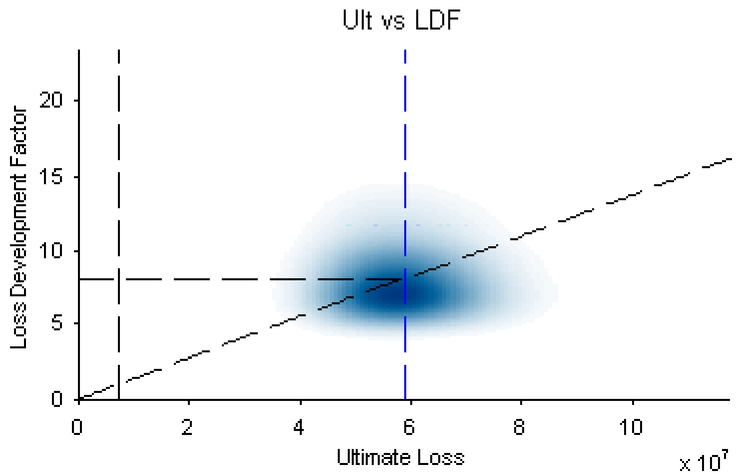


- 18 years of auto liability paid loss experience
- 24 month-to-ultimate factor
- 10,000 bootstrap replications

Filters and Smoothing

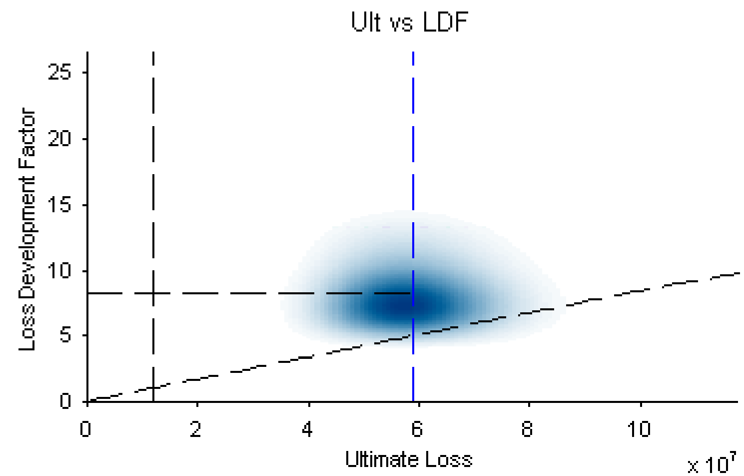
- Bootstrap densities jagged and rough
- "Low pass" filter ideal for removing high frequency noise
- Filter is essentially a moving-average
- Filter, reverse, re-filter to preserve phase
- Filtering attenuates peaks
- Filtering may introduce negative values
- Can be made into a robust smoothing technique
- Free Bonus: learn how your CD player works!
- See Hamming [3] or Numerical Recipes [6] for more details

Observed loss equal to expected



- $59M prior ultimate
- FTU = 8.09
- $7.3M observed at 24 months
- Dotted lines illustrate these quantities

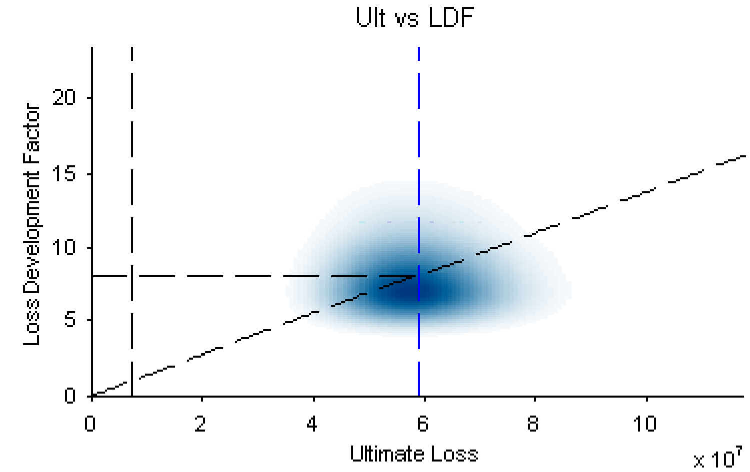
Observed loss higher than expected

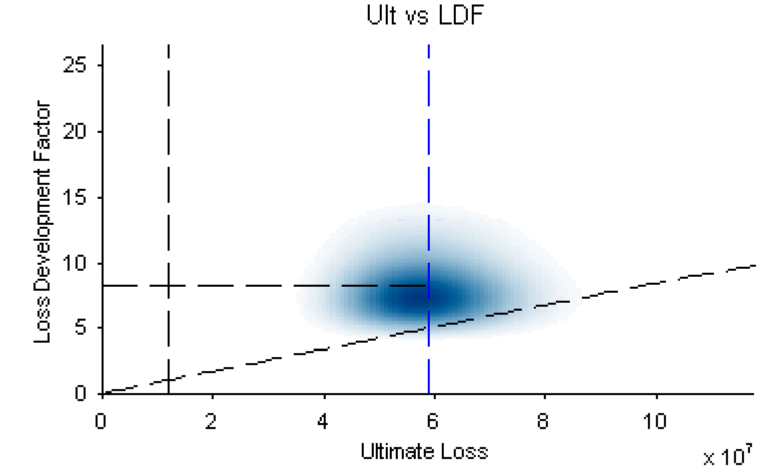


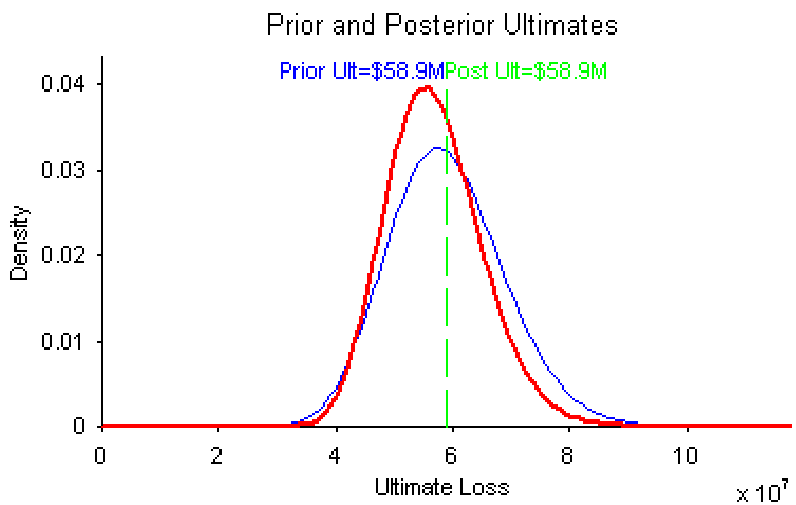
- $12M at 24 months
- 59 / 12 = 4.9 < 8.1
- Diagonal line moves *down* for *higher* observed loss
- Easy visual assessment of "significance" of observed loss

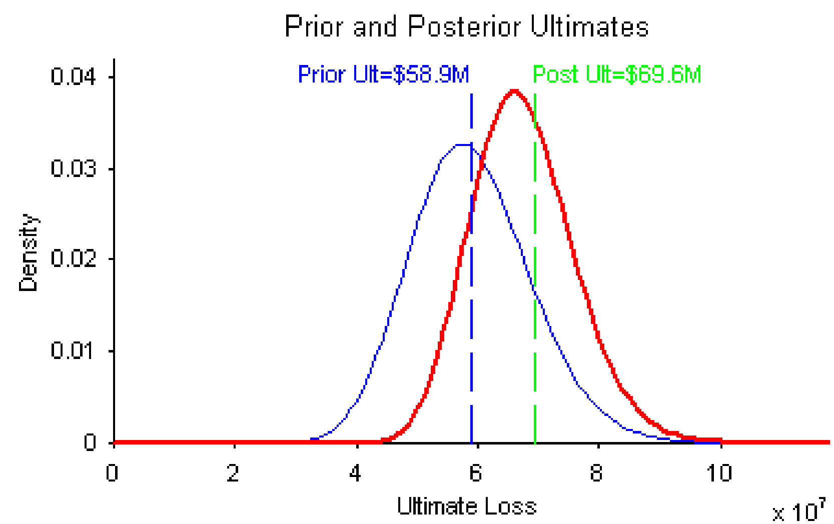
$7M at 24 mths vs. $12M at 24 mths

Posterior Distributions









Copulas and Association

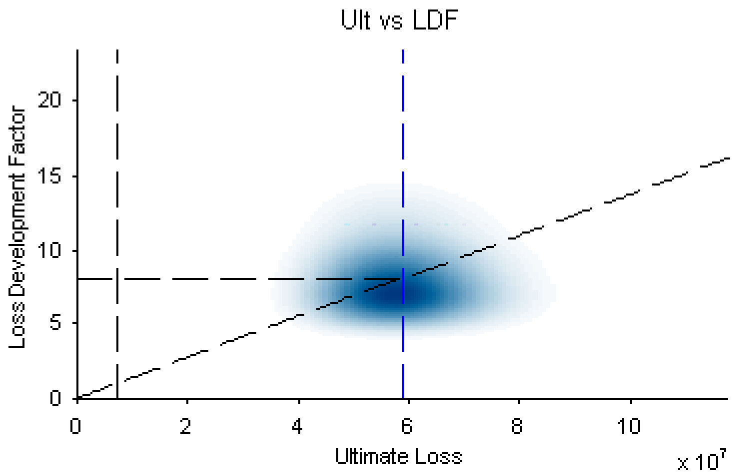
- Copulas: multivariate uniform distributions
- For a continuous bivariate distribution H there exists a unique copula C so that

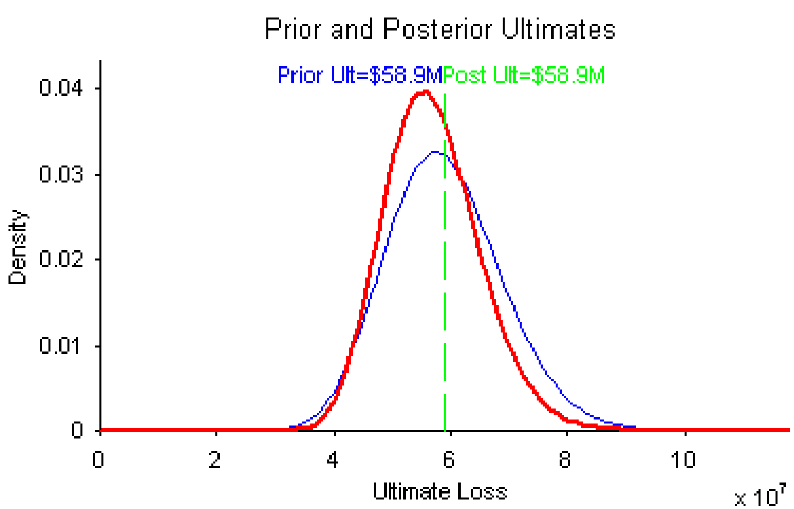
$$H(u,v) = C(H_U(u), H_V(v))$$

- $C(x,y) = xy$ corresponds to independent marginals
- Copulas capture association
- Variety of copulas available with different properties
- See Wang [9] and Frees [2]
- Non-parametric measures of association
  - Kendall’s tau and Spearman rank correlation

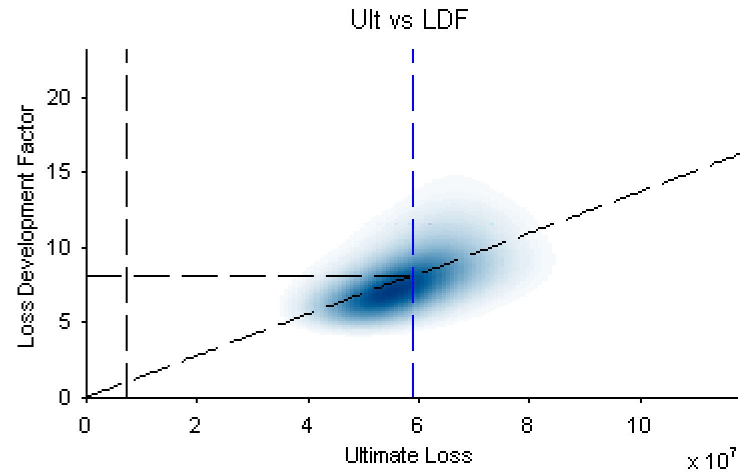
Effect of Association

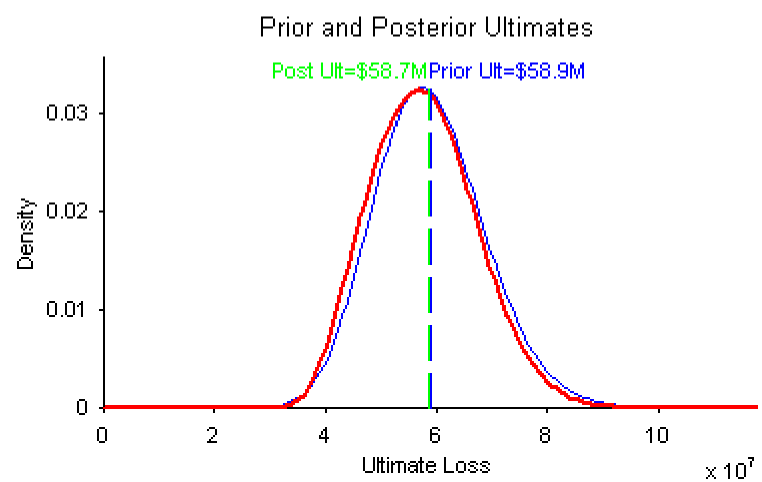
Independent





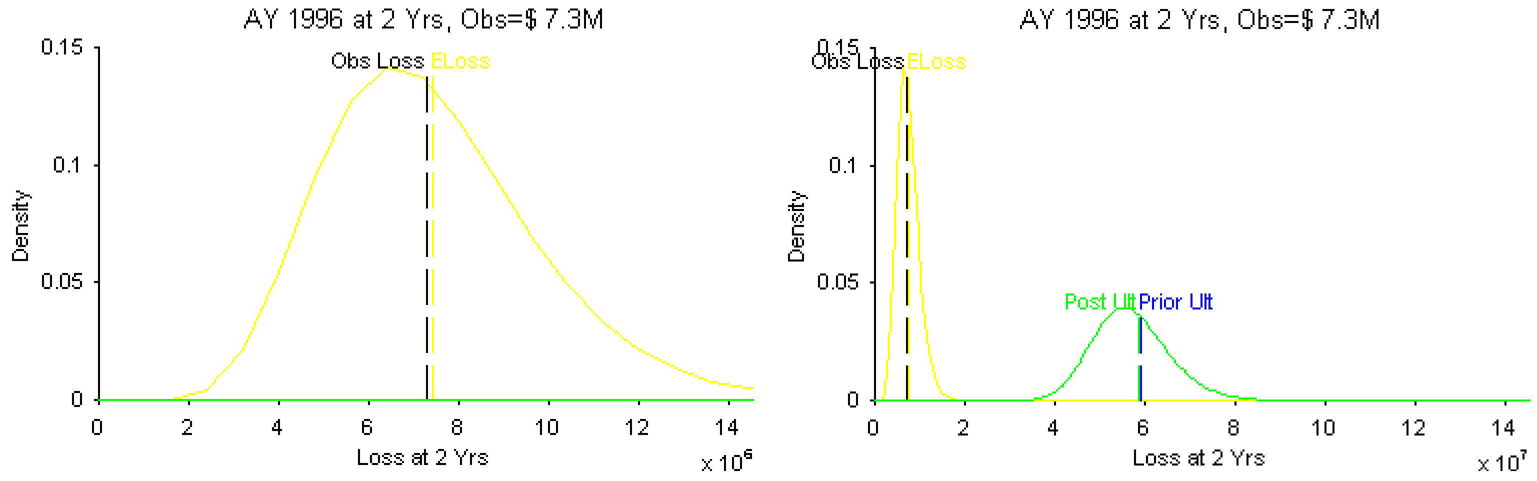
Positive Association





Frank Copula, τ=0.35

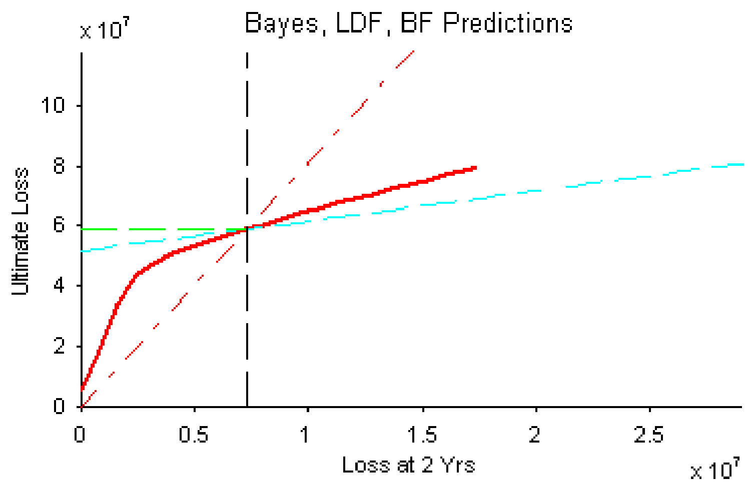
Distribution of Observed Loss



– Important for DFA
– Bootstrap method gives needed distribution for run-off conditional on observed losses
– Family of densities compatible and consistent with other model assumptions

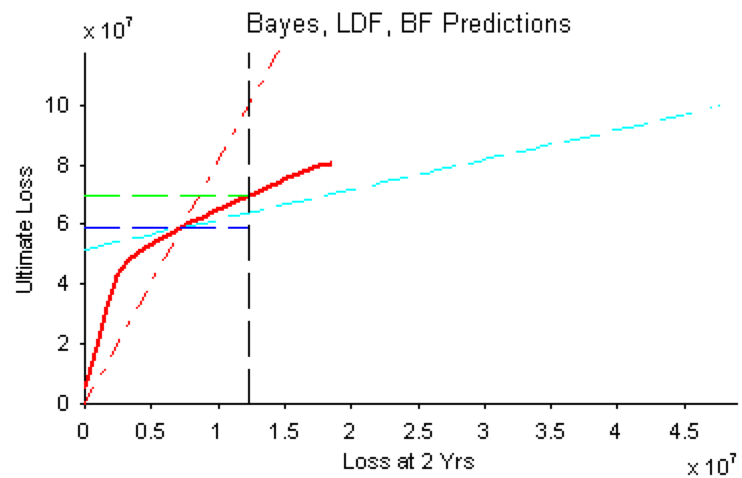
Loss Development and Credibility

Revised Ultimate



$7M at 24 mths

$59M prior ultimate



$12M at 24 mths

$59M prior ultimate

- - - - - - BF estimate of ultimate, FTU=8.1
- - - - - - Mean of posterior distribution
- - - - - - Straight development ultimate
——— Mean of posterior ultimate
- - - - - - Prior ultimate

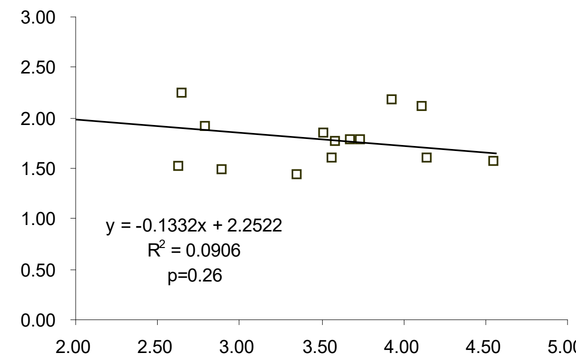
- Bayes estimate is mean of posterior distribution
- Bühlmann Credibility is best linear approximation to Bayes estimate
- Credibility of observation given by slope / FTU

Underlying Triangle

Link Ratios	2-1	3-2	4-3	5-4	6-5	7-6	8-7	9-8
1980	3.526	1.828	1.639	1.215	1.591	1.228	1.053	1.028
1981	3.747	1.758	1.827	1.328	1.263	1.210	1.104	1.063
1982	2.657	2.223	1.340	1.217	1.427	1.185	1.156	1.108
1983	2.644	1.496	1.600	1.372	1.208	1.262	1.067	1.064
1984	2.805	1.902	1.724	1.185	1.357	1.193	1.093	1.074
1985	3.691	1.758	1.437	1.440	1.143	1.346	1.136	1.043
1986	2.903	1.469	1.775	1.466	1.152	1.070	1.138	1.053
1987	3.943	2.167	1.778	1.300	1.238	1.105	1.057	1.043
1988	4.129	2.087	1.546	1.248	1.340	1.077	1.029	1.035
1989	1.956	2.706	1.532	1.541	1.454	1.190	1.063	1.031
1990	3.577	1.590	1.340	1.531	1.283	1.200	1.161	
1991	4.155	1.579	1.796	1.290	1.197	1.123		
1992	3.359	1.417	1.644	1.300	1.247			
1993	3.593	1.752	1.977	1.434				
1994	4.567	1.547	1.838					
1995	1.920	1.670						
1996	4.529							
Avg Link Ratio	3.394	1.809	1.653	1.348	1.300	1.182	1.096	1.054
FTU	27.503	**8.103**	4.478	2.709	2.010	1.546	1.308	1.193

Bolded 8.103 factor to ultimate corresponds to the FTU mentioned in slides

Correlation? 2-1 vs 3-2:

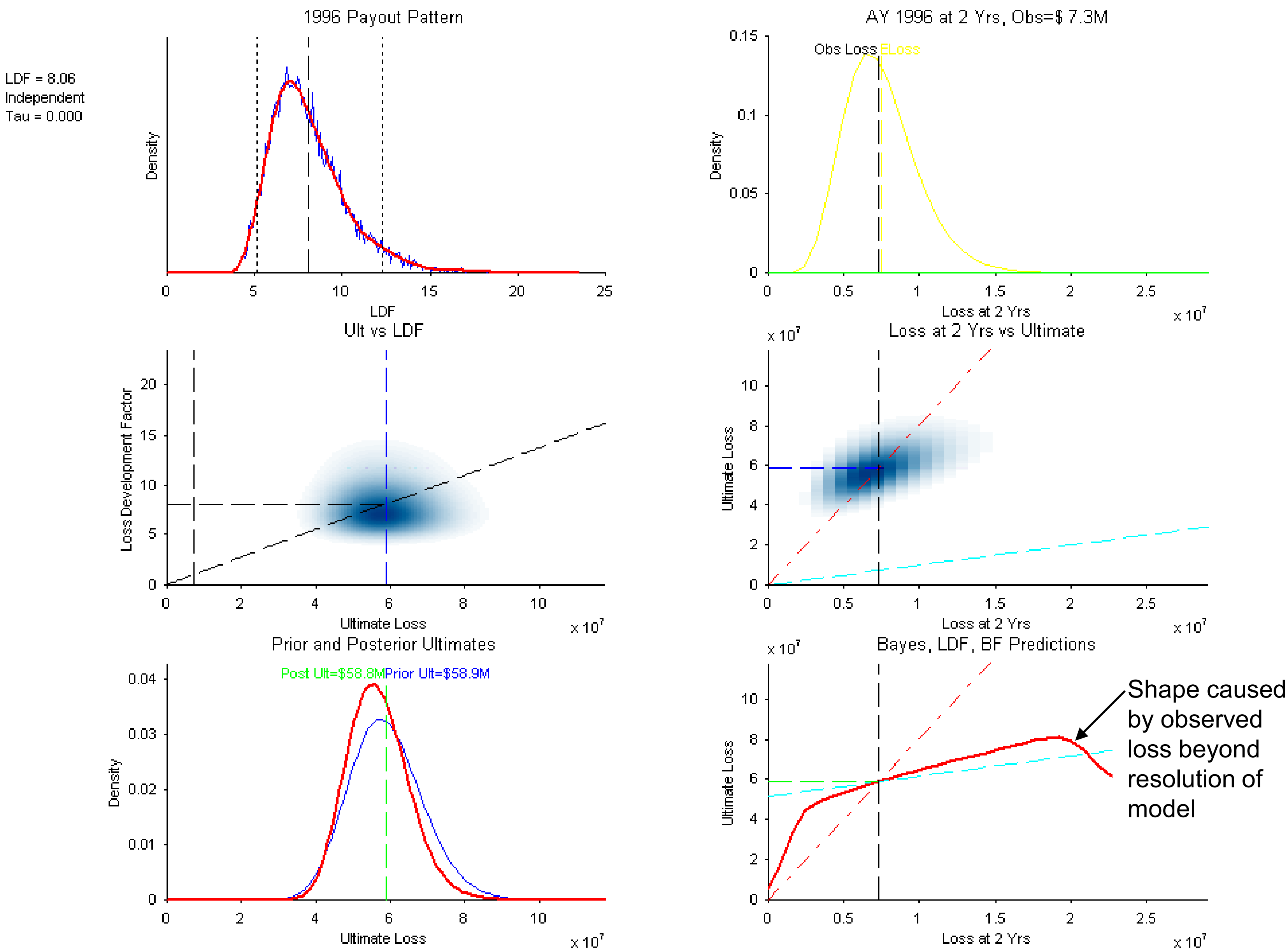


	10-9	11-10	12-11	13-12	14-13	15-14	16-15	17-16	18-17	Tail
	1.005	1.011	1.006	1.005	1.001	1.003	1.002	1.000	1.000	1.050
	1.071	1.005	1.002	1.008	1.016	1.004	1.002	1.001		1.050
	1.007	1.034	1.019	1.011	1.001	1.006	1.001			1.050
	1.025	1.022	1.008	1.002	1.005	1.002				1.050
	1.042	1.038	1.003	1.006	1.004					1.050
	1.022	1.007	1.013	1.012						1.050
	1.013	1.010	1.019							1.050
	1.044	1.007								1.050
	1.049									1.050
										1.050
										1.050
										1.050
										1.050
										1.050
										1.050
										1.050
										1.050
	1.031	1.017	1.010	1.007	1.005	1.004	1.001	1.000	1.050	
	1.132	1.098	1.080	1.069	1.061	1.056	1.052	1.050	1.050	

Source: Taylor [8], working paper

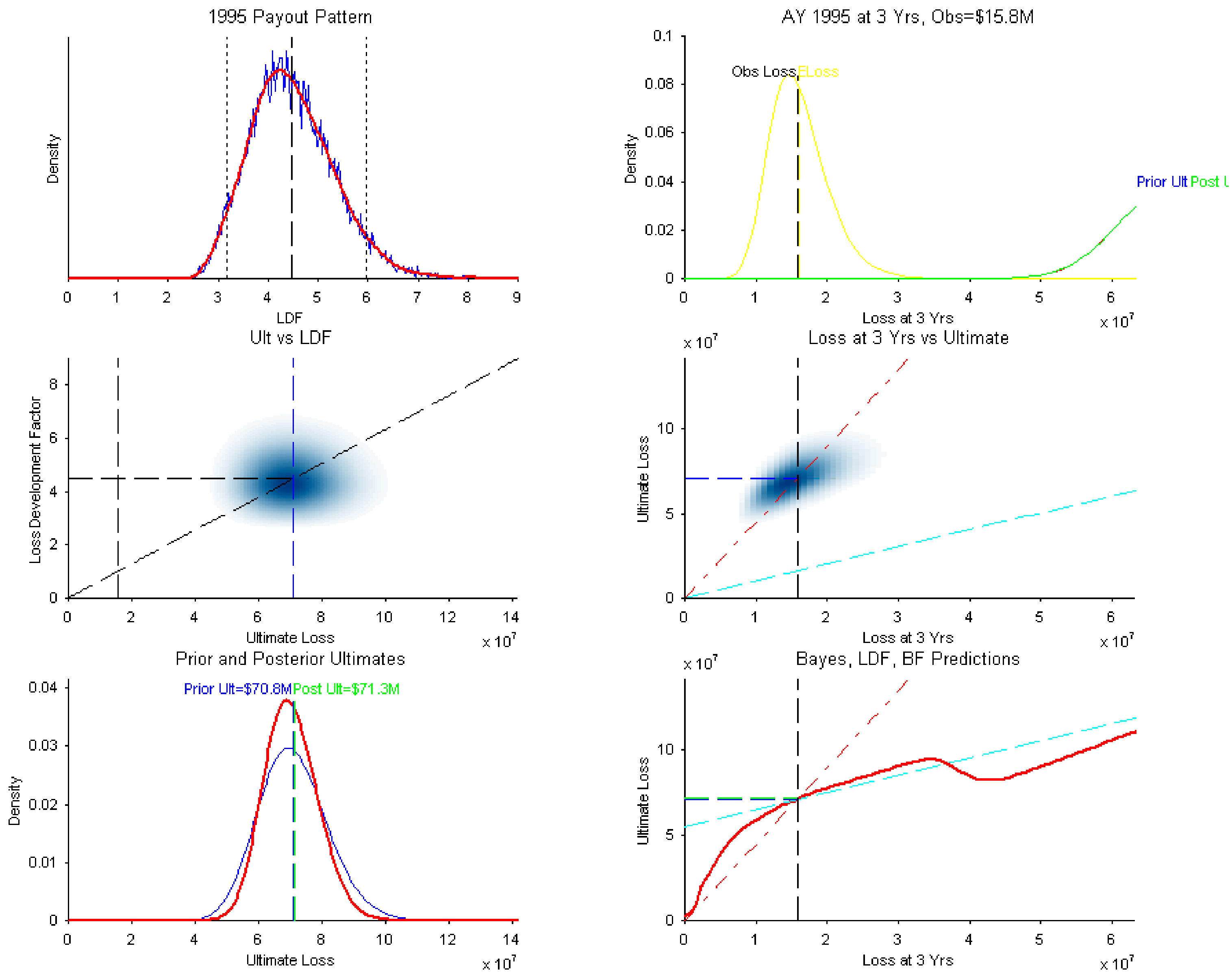
Losses "are essentially those from an Austrial-ian Auto Liability portfolio."

DFA Example, AY 1996 Loss Development Analysis (LDF view)



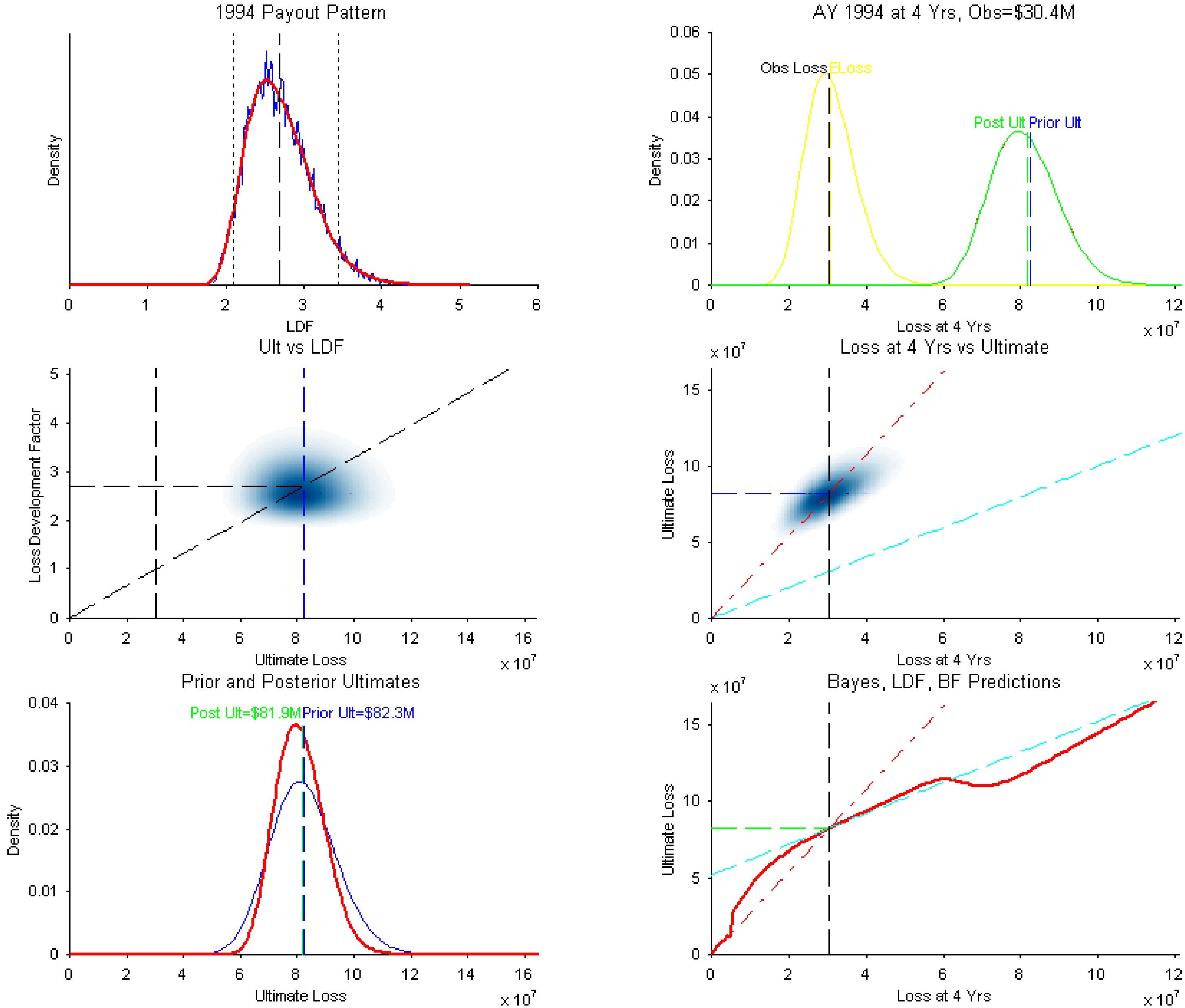
DFA Example, AY 1995 Loss Development Analysis (LDF view)

LDF = 4.48
Independent
Tau = 0.000



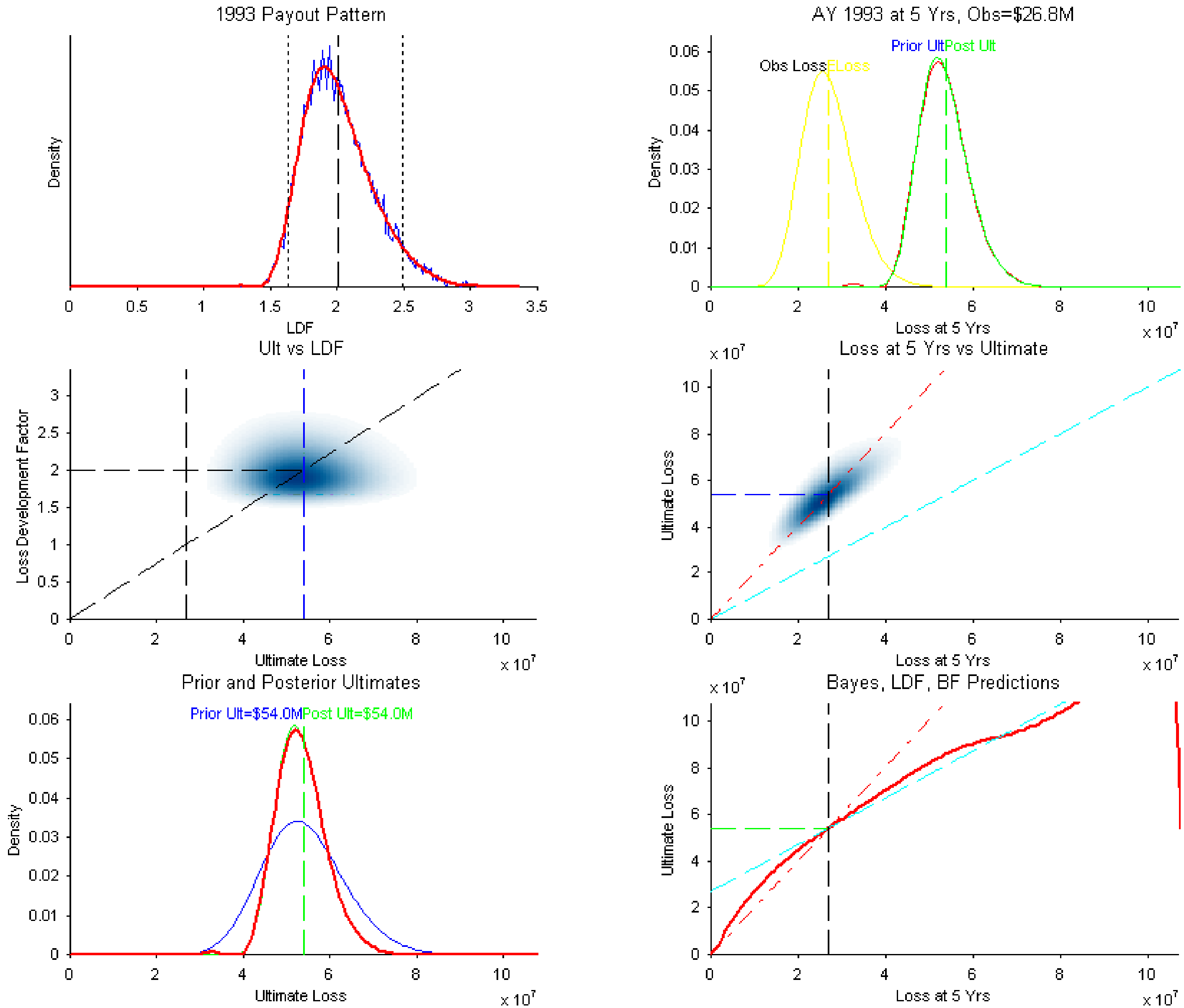
DFA Example, AY 1994 Loss Development Analysis (LDF view)

LDF = 2.70
Independent
Tau = 0.000

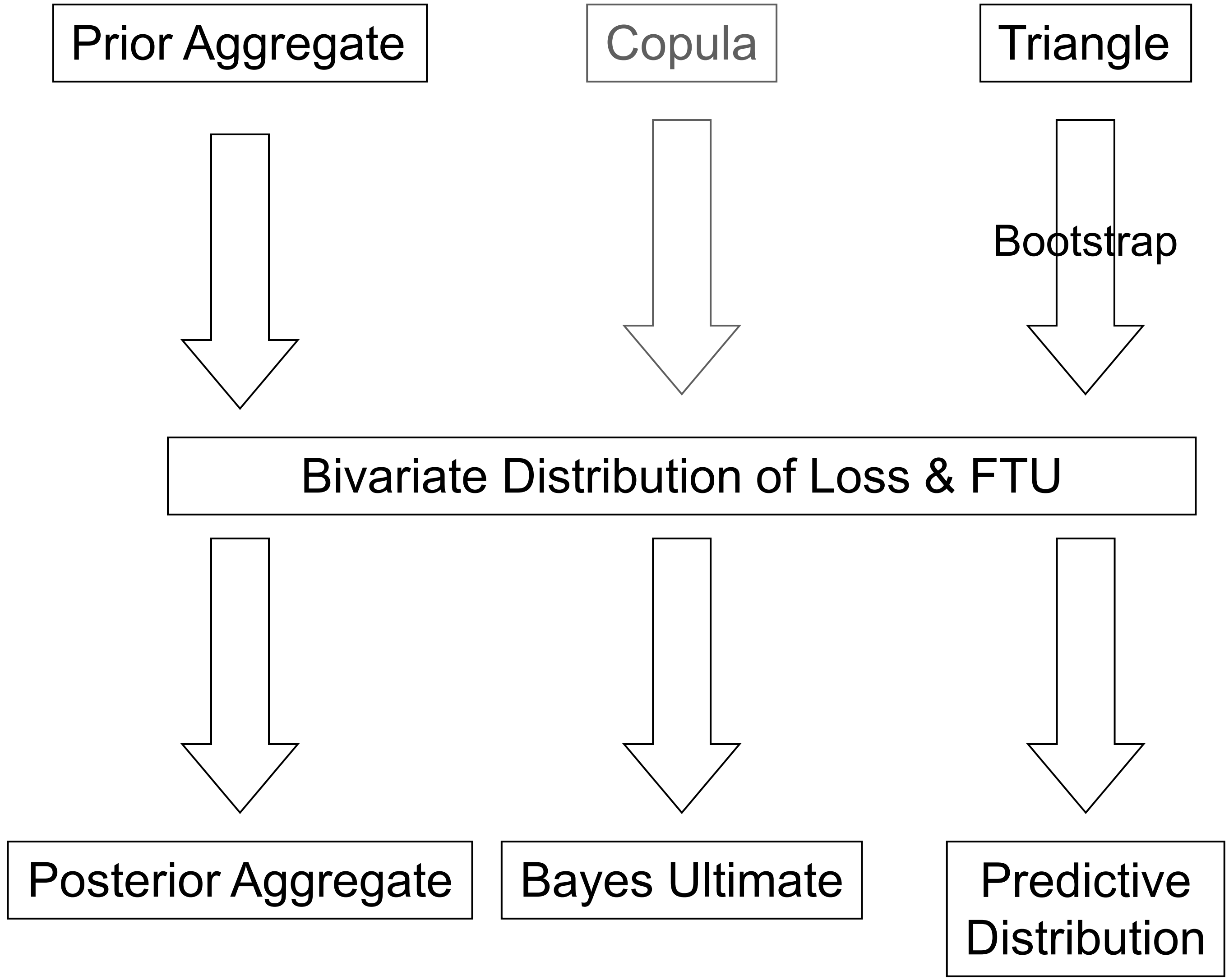


DFA Example, AY 1993 Loss Development Analysis (LDF view)

LDF = 2.01
Independent
Tau = 0.000



Summary


Prior Aggregate
Copula
Triangle
Bootstrap
Bivariate Distribution of Loss & FTU
Posterior Aggregate
Bayes Ultimate
Predictive Distribution

What have we done? What can we do?

- Bootstrap from triangle to distribution of FTU
  - Confidence intervals for FTUs
  - Distribution of discount factors
- Combine with an prior aggregate (and copula) to get bivariate distribution of ultimate and FTU
- Bayes Theorem gives posterior aggregate
  - Graphical demonstration of resolution of uncertainty
  - Applications: DFA, results analysis, reserving
- Mean of posterior gives “Bayesian” ultimates
  - Interpolate between BF and link-ratio methods
  - Reflect payout and underlying loss uncertainty in reserving process

References

[1] Efron B. and R. Tibshirani, “An Introduction to the Bootstrap,” Chapman & Hall (1993)

[2] Frees E. and E.Valdez, “Understanding Relationships Using Copulas,” NAAJ Vol. 2 No. 1 (1997)

[3] Hamming R., “Digital Filters,” 3rd Edition, Dover (1989)

[4] Intel Web Site, developer.intel.com/vtune/perflibst/spl/index.htm

[5] Ostaszewski K., and G. Rempala “Applications of Reampling Methods in Dynamic Financial Analysis,” 1998 CAS DFA Call Papers, CAS (1998)

[6] Press, W. et al., “Numerical Recipes in C,” 2nd edition, CUP (1992), www.nr.org

[7] Solomon, C., “Microsoft Office 97 Developer’s Handbook,” Microsoft Press (1997)

[8] Taylor, G., “Development of an incurred loss distribution over time,” COTOR Working Paper (1998)

[9] Wang, S., “Aggregate Loss Distributions: Convolutions and Time Dependency,” PCAS (1998), www.casact.org/coneduc/annual/98annmtg/98pcas.htm